FLOW OF A VISCOUS NEWTONIAN MEDIUM IN THE CAVITY OF A HYPERBOLIC CYLINDER AND A SINGLE-CAVITY HYPERBOLOID OF REVOLUTION

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Analytic solutions are obtained for problems of the flow of a viscous Newtonian medium within the cavity of a hyperbolic cylinder and a single-cavity hyperboloid of revolution.

It is well known that in the practice of processing various materials, in particular, polymers, we use an extrusion through fillers in which cases of convergent flows are realized. An experiment shows that it is essential to select an efficient die form, since this form also affects the properties of the products and the pressure drop needed for extrusion [1, 2]. Together with this, the rheological behavior of many processable materials is close to the behavior of a viscous incompressible fluid with a large and constant viscosity coefficient. These circumstances contribute considerably to the solution of problems for the flow of such a medium in the cavity of a hyperbolic cylinder and a single-cavity hyperboloid of revolution.

The results obtained allow us to study in detail the deformation process of a flowing medium and to estimate the pressures needed for the flow in dies of various form.

First of all, we study a problem of the two-dimensional flow of a viscous incompressible fluid in the cavity of a hyperbolic cylinder.

Let a viscous medium that satisfies the equations of motion and of incompressibility

$$\frac{\partial p}{\partial x_i} = \eta \Delta v_i, \quad \frac{\partial v_i}{\partial x_i} = 0, \tag{1}$$

fill the cavity of a hyperbolic cylinder

$$x^2 - y^2 \operatorname{tg}^2 \alpha = a_\alpha^2. \tag{2}$$

Here α is half the angle between the asymptotes of the hyperbolas formed at a section of the hyperbolic cylinder by the planes parallel to the XOY plane (Fig. 1). We call angle 2α the aperture angle of the hyperbolic cylinder. The pressures on both sides of the slot at infinity approach constant values that differ in magnitude. The medium flows because of the pressure difference, and the flow velocity becomes zero on surface (2) due to the adhesion characteristic of viscous media. We determine the velocity fields, the deformation velocities, and the pressures in the flowing medium.

To describe the process under study it is convenient to use the λ, μ, and z coordinates of the elliptical cylinder [3] (Fig. 1), which are related to the x, y, and z coordinates by the equations

$$x^2 = a_0^2 (1+\lambda)(1+\mu); \; y^2 = -a_0^2 \lambda\mu; \; z = z. \tag{3}$$

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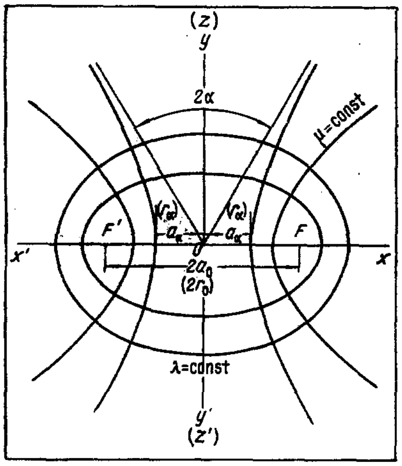

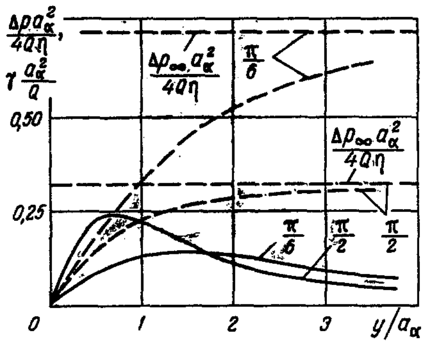

Fig. 1 Fig. 2

Fig. 1. Orthogonal system of cofocal ellipses and hyperbolas with focuses F and F'. These curves generate coordinate surfaces in the following coordinate systems: elliptical cylinder (when shifting perpendicular to the figure plane); flattened ellipsoid of revolution [when revolving around the axis (z), (z')].

Fig. 2. Variation of Δp (dashed lines) and γ (solid lines) with coordinate in the symmetry plane of a hyperbolic cylinder.

The coordinates of the line here are cofocal ellipses, cofocal hyperbolas, and planes parallel between themselves. Here λ, μ, and z vary at the following limits:

$$0 \leqslant \lambda \leqslant \infty; \ -1 \leqslant \mu \leqslant 0; \ -\infty \leqslant z \leqslant \infty.$$

We note that in this coordinate system the equation of the plane y = 0 between the lines $x = -a_0$ and $x = a_0$ takes the form $\lambda = 0$; the equation of the same plane beyond the limits of the given interval takes the form $\mu = 0$, and the equation of the plane x = 0 takes the form $\mu = -1$. The Lamé coefficients for the coordinate system selected are determined by the equations

$$H_\lambda = \frac{a_0}{2} \sqrt{\frac{\lambda - \mu}{\lambda(1+\lambda)}}; \quad H_\mu = \frac{a_0}{2} \sqrt{\frac{\lambda - \mu}{-\mu(1+\mu)}}; \quad H_z = 1. \tag{4}$$

It is evident that in the problem under study the quantities of interest to us do not depend on z, and the velocity of the vector along the x axis is equal to zero. Thus, in correspondence with the theory of orthogonal curvilinear coordinates (see [4], for example), the nonzero components of the viscous stress tensor τ_{ik} in the λ and μ coordinates take the form

$$\begin{gathered}
\tau_{\lambda\lambda} = \frac{2\eta}{H_\lambda}\left(\frac{\partial v_\lambda}{\partial \lambda} + \frac{v_\mu}{H_\mu} \cdot \frac{\partial H_\lambda}{\partial \mu}\right); \\
\tau_{\mu\mu} = \frac{2\eta}{H_\mu}\left(\frac{\partial v_\mu}{\partial \mu} + \frac{v_\lambda}{H_\lambda} \cdot \frac{\partial H_\mu}{\partial \lambda}\right); \\
\tau_{\lambda\mu} = \eta\left\{\frac{1}{H_\mu} \cdot \frac{\partial v_\lambda}{\partial \mu} + \frac{1}{H_\lambda} \cdot \frac{\partial v_\mu}{\partial \lambda} - \frac{1}{H_\lambda H_\mu}\left[v_\lambda \frac{\partial H_\lambda}{\partial \mu} + v_\mu \frac{\partial H_\mu}{\partial \lambda}\right]\right\}.
\end{gathered} \tag{5}$$

The equations of motion (1) can be written in terms of the components of the stress tensor (5) in the assumed coordinate system as follows [4]:

$$\frac{\partial p}{\partial\lambda}=\frac{1}{H_\mu}\frac{\partial H_\mu}{\partial\lambda}(\tau_{\lambda\lambda}-\tau_{\mu\mu})+\frac{\partial\tau_{\lambda\lambda}}{\partial\lambda}+\frac{2}{H_\mu}\cdot\frac{\partial H_\lambda}{\partial\mu}\tau_{\lambda\mu}+\frac{H_\lambda}{H_\mu}\cdot\frac{\partial\tau_{\lambda\mu}}{\partial\mu};$$
$$\frac{\partial p}{\partial\mu}=\frac{1}{H_\lambda}\cdot\frac{\partial H_\lambda}{\partial\mu}(\tau_{\mu\mu}-\tau_{\lambda\lambda})+\frac{\partial\tau_{\mu\mu}}{\partial\mu}+\frac{H_\mu}{H_\lambda}\cdot\frac{\partial\tau_{\lambda\mu}}{\partial\lambda}+\frac{2}{H_\lambda}\cdot\frac{\partial H_\mu}{\partial\lambda}\tau_{\lambda\mu}. \tag{6}$$

The equation of incompressibility in the given case reduces to the equation

$$\frac{\partial(H_\mu v_\lambda)}{\partial\lambda}+\frac{\partial(H_\lambda v_\mu)}{\partial\mu}=0. \tag{7}$$

We seek the solution for the flow velocity in the form

$$v_\lambda=v_\lambda(\lambda,\ \mu),\ v_\mu=0, \tag{8}$$

i.e., we assume that the lines of flow in the process under study are hyperbolas of the family of the coordinate lines. Thus, after substituting (4) and (5) into (6) and taking (7) and (8) into account, we have

$$\frac{\partial p}{\partial\lambda}=\frac{\eta}{a_0}\cdot\frac{1}{\sqrt{\lambda(1+\lambda)(\lambda-\mu)}}\left[v_\lambda\frac{(\lambda+\mu)(1+\mu)+\mu(1+\lambda)}{2(\lambda-\mu)^2}-\frac{\partial v_\lambda}{\partial\mu}(1+2\mu)-\frac{\partial^2 v_\lambda}{\partial\mu^2}2\mu(1+\mu)\right]; \tag{9.1}$$

$$\frac{\partial p}{\partial\mu}=\frac{\eta}{a_0}\cdot\frac{\sqrt{\lambda(1+\lambda)}}{(\lambda-\mu)\sqrt{\lambda-\mu}}\left(2\frac{\partial v_\lambda}{\partial\mu}-v_\lambda\frac{1}{\lambda-\mu}\right). \tag{9.2}$$

After differentiating the first of these equations with respect to μ and the second with respect to λ and subtracting one from the other, we eliminate the pressure and obtain the equation for the velocity, which we do not present here. By immediately substituting this equation into (9) and conducting the procedure indicated, we can verify that the general solution for the flow velocity in the case under study takes the form

$$v_\lambda=\frac{A\mu+B}{\sqrt{\lambda-\mu}}. \tag{10}$$

Here A and B are constants. We use the following two conditions to determine the constants. First of all, the velocity is equal to zero on the surface of the hyperbolic cylinder. Consequently, for $\mu = \mu_\alpha$, according to (10), we have

$$A\mu_\alpha+B=0.$$

In accordance with (2) and (3) $\tan^2\alpha = (1+\mu_\alpha)/-\mu_\alpha$, i.e.,

$$\mu_\alpha=-\cos^2\alpha \text{ and } B-A\cos^2\alpha=0. \tag{11}$$

In addition, we use the quantity of the second liquid flow rate per unit of the slot. For a cylinder corresponding to the one chosen, $\mu_\alpha=\text{const},\ Q=2\int_0^{\mu_\alpha}H_\mu v_\lambda d\mu$. When calculating Q, if we consider that $a_\alpha = a_0\sin\alpha$ [see (3)], then

$$Q=\frac{a_\alpha}{\sin\alpha}\left[2\alpha B-\left(\alpha+\frac{1}{2}\sin 2\alpha\right)A\right]. \tag{12}$$

After substituting the calculated values of A and B from (11) and (12) into (10), we arrive at the following equation for the flow velocity:

$$v_\lambda=-\frac{2Q}{a_\alpha}\cdot\frac{\sin\alpha}{(\sin 2\alpha-2\alpha\cos 2\alpha)}\cdot\frac{\mu+\cos^2\alpha}{\sqrt{\lambda-\mu}}. \tag{13}$$

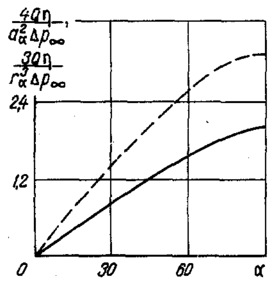

Fig. 3

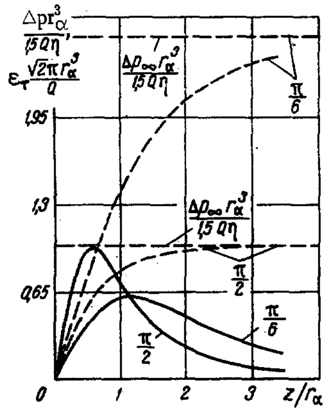

Fig. 4

Fig. 3. Second flow rate of material versus α deg(dashed curve refers to hyperbolic cylinders, and solid lines show hyperboloids of revolution).

Fig. 4. Variation of Δp (dashed lines) and ε_T (solid lines) along the symmetry axis of the hyperboloid of revolution.

(The velocity is distinguished only by the sign for points symmetric with respect to the slot.) In particular, for $\alpha = \pi/2$ this equation yields the solution of the problem for the flow of a viscous incompressible fluid into a slot of infinite width. The solution is obtained in [5, 6]. In addition, it is not difficult to obtain from this solution known solutions of problems for the flow of a viscous medium between two parallel planes and in a plane convergent channel. At the same time, the solution that was found allows us to study the properties and characteristics of the velocity field, the stresses, and the deformation velocities established in a material that flows in hyperbolic cylinders with any aperture angles in order to compare the possible variants.

We also present the equation for the principal shear velocity:

$$\gamma = \frac{2Q}{a_\alpha^2} \cdot \frac{\sin^2\alpha \sqrt{\mu\lambda\cos 2\alpha + (1+\lambda+\mu)\cos^4\alpha}}{(\sin 2\alpha - 2\alpha\cos 2\alpha)(\lambda-\mu)^{3/2}}. \tag{14}$$

We calculate the pressure. It follows from the conditions of the problem that the pressure is determined with accuracy up to an arbitrary term, since the flow of the medium is caused by the difference in the pressures on both sides of the slot. We see from Eq. (9.2) that the pressure in the plane of the slot is constant. Thus, the difference between the pressure at any point of the upper part of the cylinder and the slot (the pressure drop) can be obtained by integrating (9.1) over λ from 0 to λ after substituting (13) into (9.1). Here

$$\Delta p = \frac{4Q\eta}{a_\alpha^2} \cdot \frac{\sin^2\alpha}{(\sin 2\alpha - 2\alpha\cos 2\alpha)} \cdot \frac{\sqrt{\lambda(1+\lambda)}}{\lambda-\mu}. \tag{15}$$

If $\lambda\to\infty$, Δp, independently of μ, approaches the constant

$$\Delta p_\infty = \frac{4Q\eta}{a_\alpha^2} \cdot \frac{\sin^2\alpha}{(\sin 2\alpha - 2\alpha\cos 2\alpha)}. \tag{16}$$

The equation given establishes a relation between the quantity of the flow rate of the medium and the limit pressure drop.

Figures 2 and 3 illustrate some of the results obtained (they refer to the upper part of the hyperbolic cylinder). Figure 2 shows the variation of Δp along the y axis for two of the possible configurations. It is evident that in both cases a significant variation in the pressure drop occurs only in immediate proximity to the slot. At a distance from the slot equal to two or three times its width, the value Δp is very close to the limiting value. The result allows us to estimate the applicability of the obtained solutions to those cases, when, as always happens, the flowing medium occupies a bounded region. Now we can confirm that the solutions presented are suitable for those specific processes in which the medium fills a region that exceeds the width of the slot by several times its dimensions. We also draw attention to the fact that Δp decreases with an increasing aperture angle for an equal flow rate and an identical value of a_α. Dependences of the principal shear velocity on the coordinate are also shown in the same figure. Their difference from each other for $\alpha = \pi/6$ and $\alpha = \pi/2$ also emphasizes the important role of the form of the device in which the flow occurs.

Figure 3 shows the variation of the value of the second flow rate of the material with an increase of α.

We proceed to the case of a flow in the cavity of a single-cavity hyperboloid of revolution. We assume in the present case that the viscous incompressible medium fills such a hyperboloid, and the difference in pressures at infinity on both sides of the throat circumference guarantees the flow of the medium. As before, we consider that the material adheres to the walls of the hyperboloid. The problem reduces to the solution of the system of equations (1). We must note that a similar problem is studied in [7], but errors were made in the calculations.

It is well known that the solution of this or any other hydrodynamic problem can be considerably facilitated by the efficient choice of a coordinate system. In the present study we use the coordinates of a flattened ellipsoid of revolution (Fig. 1). In the three-dimensional case under study the third coordinate, φ, is naturally added to λ and μ, where λ, μ, and φ are related to the cylindrical coordinates z, r, and φ by the equations

$$z^2 = -r_0^2\lambda\mu;\quad r^2 = r_0^2(1+\lambda)(1+\mu);\quad \varphi = \varphi \tag{17}$$

and vary at the limits

$$0 \leqslant \lambda \leqslant \infty;\ -1 \leqslant \mu \leqslant 0;\ 0 \leqslant \varphi \leqslant 2\pi.$$

The equation of the plane z = 0 beyond the limits of the circumference with radius r_0 takes the form $\mu = 0$; the equation of the same plane within these limits takes the form $\lambda = 0$, and the equation of the z axis takes the form $\mu = -1$. We also present the Lamé coefficients for the given system of coordinates

$$H_\lambda = \frac{r_0}{2}\sqrt{\frac{\lambda-\mu}{\lambda(1+\lambda)}};\quad H_\mu = \frac{r_0}{2}\sqrt{\frac{\lambda-\mu}{-\mu(1+\mu)}};\quad H_\varphi = r_0\sqrt{(1+\lambda)(1+\mu)}. \tag{18}$$

We introduce the aperture angle 2α for each hyperboloid similar to the case of a hyperbolic cylinder studied above. According to (17),

$$\mathrm{tg}^2\alpha = \frac{1+\mu_\alpha}{-\mu_\alpha},\ \text{a}\ r_\alpha = r_0\sin\alpha.$$

Since we solved the problem by a method completely analogous to that described for the preceding case, we omit all the intervening calculations and present only the results obtained.

It follows from the solution of the problem that the lines of flow in the given process are hyperbolas obtained at the cross section of the surfaces μ = const and the planes φ = const. Here the flow velocity takes the form

$$v_\lambda = -\frac{3Q}{2\pi r_\alpha^2}\cdot\frac{\sin^2\alpha}{(1-3\cos^2\alpha+2\cos^3\alpha)}\cdot\frac{\mu+\cos^2\alpha}{\sqrt{(1+\lambda)(\lambda-\mu)}}. \tag{19}$$

The particular case $\alpha = \pi/2$ yields the solution of the problem for the flow into the circular hole which was obtained in [5]. We note that attempts to determine the flow velocities for this case in [8, 9] did not lead to a final exact result.

The equation for the deformation velocity of the resulting shear at points of the symmetry axis takes the form

$$\varepsilon_\tau = \frac{3Q}{\sqrt{2}\pi r_\alpha^3} \cdot \frac{\sin^5\alpha}{(1-3\cos^2\alpha+2\cos^3\alpha)} \cdot \frac{\sqrt{\lambda}}{(1+\lambda)^2}. \qquad (20)$$

The pressure drop for the upper part of the hyperboloid in the coordinate system used is determined by the equation

$$\Delta p = \frac{3Q\eta}{\pi r_\alpha^3} \cdot \frac{\sin^3\alpha}{(1-3\cos^2\alpha+2\cos^3\alpha)} \left(\frac{\sqrt{\lambda}}{\lambda-\mu} + \operatorname{arctg}\sqrt{\lambda}\right), \qquad (21)$$

$$\Delta p_\infty = \frac{3Q\eta}{2r_\alpha^3} \cdot \frac{\sin^3\alpha}{(1-3\cos^2\alpha+2\cos^3\alpha)}. \qquad (22)$$

Graphs obtained on the basis of (20) and (21) for the variation of Δp and ε_τ are presented with the coordinate along the symmetry axis in Fig. 4. The dependency of the second flow rate on α that was constructed in (22) is shown in Fig. 3.

NOTATION

p, pressure; η, viscosity; v_i, velocity components; a_α, half-width of hyperbolic cylinder slot; r_α, radius of throat circumference of single-cavity hyperboloid of revolution; 2α, aperture angles of hyperbolic cylinder and hyperboloid of revolution; λ, μ, and z, coordinates of elliptical cylinder; λ, μ, and φ, coordinates of flattened ellipsoid of revolution; H_λ, H_μ, H_z, and H_φ, Lamé coefficients; τ_{ik}, components of viscous stress tensor; Q, second liquid flow rate per unit length of slot for hyperbolic cylinder and volume second flow rate of liquid for hyperboloid of revolution; γ, principal shear velocity; ε_τ, strain rate of resulting shear; Δp, pressure drop; Δp_∞, limit pressure drop.

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